

UTT-07-95

February 7, 2008

Collective Coordinates in String Theory

Willy Fischler*, Sonia Paban*, Moshe Rozali*

*Theory Group, Department of Physics
University of Texas, Austin, Texas 78712*

Abstract

The emergence of violations of conformal invariance in the form of non-local operators in the two-dimensional action describing solitons inevitably leads to the introduction of collective coordinates as two dimensional “wormhole parameters”.

*Research supported in part by the Robert A. Welch Foundation and NSF Grant PHY 9009850

1 Introduction

This paper addresses the problems associated with describing scattering of a spacetime soliton in the context of string theory. More specifically we will present a way to take into account the changes in the soliton state due to the scattering. These may include recoil as well as changes in the various charges of the soliton. Our interest in this question goes beyond the scattering problem itself. Rather, we wish to address the conceptual puzzle of how to properly overlap different conformal field theories.

The organization of the paper is as follows: We first review some general properties of solitons (for a complete discussion, including references, see [1, 2]). In particular we show how these scattering questions are resolved in pointlike theories. We explain why the scattering off a monopole is problematic in string theory. We then show how in string theory the presence of monopoles induces violations of conformal invariance in the form of bilocal operators on the worldsheet. One is led then to introduce wormhole parameters that turn out to account for the finite number of degrees of freedom associated with the recoil or change of charge.

For the sake of clarity we will focus on monopoles, but the conclusions also apply to other extended objects. Monopoles are time-independent, finite energy, stable solutions of classical field equations for weakly coupled systems. The weak coupling ensures that these monopoles are heavy and therefore can be reasonably well described by classical physics.

These solutions break in general some symmetries of the Lagrangian. For example, the location in space of the monopole breaks translational invariance: obviously, the energy of the soliton is independent of its center of mass location. This fact implies the existence of normalizable “zero modes” when quantizing the theory in a monopole background. These modes are eigenfunc-

tions of the differential operator describing linearized fluctuations around the monopole. The role of these modes in pointlike theories is crucial in calculating the scattering off a monopole when the monopole state changes. These are the scattering amplitudes we focus on in this paper.

In pointlike theories such amplitudes are calculated using second quantization (field theory). The essential ingredient in the calculation is a change of variables that trades each zero mode for a collective coordinate. In the example of the recoiling monopole, there are D translational zero modes that are exchanged for D spatial coordinates of the monopole's center of mass (in $D+1$ dimensions).

In transforming to these new coordinates care has to be exercised in calculating the Jacobian of the transformation. In a path integral formulation this involves a straightforward but tedious Fadeev-Popov procedure.

After having changed to these new variables, we are still dealing with a degenerate perturbation theory. Indeed, monopoles located in different positions have the same energy. It is important then to use the right basis in the degenerate subspace in order to avoid vanishing energy denominators. (In diagrammatic language these energy denominators appear when there are zero modes in intermediate states, which then lead to spacetime infrared divergences). The correct variables to use are the components of the momentum conjugate to the center of mass position.

This summarizes the various steps that are necessary in pointlike theories in order to calculate scattering amplitudes.

In string theory this simple scattering presents some problems, since we do not have a workable second quantized formulation. In a first quantized string theory one represents a heavy monopole (positioned for example at the origin) by a two-dimensional conformal field theory. In general the monopoles initial

and final states in a scattering process are different. This requires interpolation between different conformal field theories, which in turn would seem to require a second quantized version of string theory.

However, as we show next, the recoil or change of charge of the monopole in a scattering process can be described in a first quantized formulation.

2 Worldsheet wormholes

We will treat the monopole as a two dimensional conformal field theory described by an action:

$$I = \int d^2z \sum_i g_i(x) \mathcal{O}_i(\partial_\alpha x, \mathcal{J}_\alpha). \quad (2.1)$$

$g_i(x)$ are functionals of the fields $x_\mu(z, z^*)$, $\mu = 0 \dots, D$, and satisfy the β -function equations:

$$\beta(g_i) = 0 \quad (2.2)$$

\mathcal{O}_i are functionals of $\partial_\alpha x$ and possibly other Kac-Moody currents \mathcal{J}_α . Specific actions describing monopoles and other solitons can be found for example in [3].

The proper treatment of the recoil of the soliton (change of conformal field theory) emerges through the existence of new violations of conformal invariance in the conformal field theory (2.1). These anomalies first appear at one loop and arise from a long thin handle attached onto a sphere. In the representation of higher genus surfaces as the plane with pairs of discs cut out and identified (Figure 1), the relevant limit in moduli space is shrinking one pair of discs to zero while keeping their distance fixed. The effects of such a handle in the degeneration limit can be represented by bilocal operators inserted on a surface of lower genus.

The existence of these world sheet divergences can be anticipated from the spacetime interpretation. Indeed, these divergences are due to the propagation in the degenerate handle of spacetime normalizable zero modes, which are separated from the continuum by a gap. As mentioned above this is related to vanishing energy denominators in a degenerate perturbation theory, causing spacetime infrared divergences. In the worldsheet language the divergences are manifest as worldsheet cut-off dependence, or in other words conformal anomalies.

To regulate the violation of conformal invariance we introduce a short distance cut-off ϵ on the worldsheet. To properly extract the worldsheet cut-off dependence we use a formalism developed by J. Polchinski [4]. Given two closed Riemann surfaces Σ_1 and Σ_2 , one can form a new surface Σ by joining Σ_1 and Σ_2 with a long thin tube (figure 2). One can then relate S-matrix elements on Σ to those on Σ_1 and Σ_2 , as given by the following expression:

$$\begin{aligned} \int dm_\Sigma \left\langle \prod_i \int d^2 z_i V_i(z_i) \prod_k B_k(z_k) \right\rangle_\Sigma &= \\ = \sum_a \int \frac{dq}{q^{h_a-1}} \frac{d\bar{q}}{\bar{q}^{\tilde{h}_a-1}} \int d^2 z_1 \int d^2 z_2 \int dm_{\Sigma_1 \oplus \Sigma_2} & \quad (2.3) \\ \left\langle \prod_i \int d^2 z_i V_i(z_i) \prod_k \hat{B}_k(z_k) \hat{b}_{-1} \hat{b}_{-1} \phi_a(z_1) \hat{b}_1 \hat{b}_1 \phi_a(z_2) \right\rangle_{\Sigma_1 \oplus \Sigma_2} \end{aligned}$$

where ϕ_a are operators corresponding to a complete set of L_0 and \tilde{L}_0 eigenstates

$$\begin{aligned} L_0 |\phi_a\rangle &= h_a |\phi_a\rangle \\ \tilde{L}_0 |\phi_a\rangle &= \tilde{h}_a |\phi_a\rangle . \end{aligned} \quad (2.4)$$

q is the radius of the discs ($arg(q)$ is an identification angle). $\int dm_\sigma$ is the integration over the moduli of the respective surface σ weighted by appropriate ghost insertions B_k or \hat{B}_k [4].

In the problem at hand the two surfaces Σ_1 and Σ_2 are the same surface, and the $q \rightarrow 0$ limit corresponds to a long and thin wormhole attached to the

surface. It is clear from (2.2) that possible logarithmic divergences (and hence cut-off dependence) for small q can only be generated by intermediate states satisfying $h_a = \tilde{h}_a = 0$. However, not all such states contribute to a divergence. The zero momentum components of the usual massless string states do not lead to divergences since they are of zero measure in the sum over states in (2.2). Therefore the only possible divergent contribution can come from discrete zero eigenstates of L_0 and \tilde{L}_0 .

Such modes exist in the spectrum of L_0 and \tilde{L}_0 . This is the spectrum of linearized fluctuations in the monopole's background. As discussed above, this spectrum has zero eigenmodes. These modes are normalizable in space and therefore are discrete. If working in a finite (but large) time interval T , they are a discrete part of the spectrum of L_0 and \tilde{L}_0 (not just the time independent part), thus enabling us to properly extract the divergent contribution to (2.2).

For the sake of clarity we'll concentrate on the the zero modes associated with translations in space. The operators associated with these modes are:

$$\varphi_k = N c\bar{c} \frac{\partial g_i(x)}{\partial x} \mathcal{O}_i(\partial_\alpha x, \mathcal{J}_\alpha) \quad (2.5)$$

where N is a normalization constant. Using the Heisenberg equations of motion one can write:

$$\varphi_\kappa = N c\bar{c} \partial_\alpha j_\kappa^\alpha \quad (2.6)$$

where j_k^α are the two-dimensional Nöether currents associated with translations in space:

$$x(z, z^*) \rightarrow x(z, z^*) + q^\kappa$$

The normalization constant is computed in the appendix and is found to be

$$N^2 = -\frac{4\pi}{\kappa} \log \epsilon. \quad (2.7)$$

Where ϵ is the cut-off introduced earlier, and the constant κ is defined from

the following expectation value on the sphere;

$$\langle j^i(z) j^j(0) \rangle = \frac{\kappa \delta^{ij}}{z^2}. \quad (2.8)$$

The appearance of a divergent normalization constant is not surprising since in the continuum limit the operators $\partial_\alpha j_i^\alpha$ are BRST null and therefore have zero norm.

Using the factorization formula (2.2), we see therefore that the cut-off dependence due to a worldsheet wormhole can be summarized by the following bilocal insertion:

$$\sum_i \frac{1}{\kappa T} (4\pi \log \epsilon)^2 \int d^2 z_1 \partial_\alpha j_i^\alpha(z_1) \int d^2 z_2 \partial_\alpha j_i^\alpha(z_2)$$

where T is the total (target) time introduced earlier.

The effect of a dilute gas of wormholes on the sphere exponentiates, and can therefore be summarized by a change in the action (2.1) on the sphere

$$\Delta I = \frac{1}{\kappa T} (4\pi \log \epsilon)^2 \sum_i \int d^2 z_1 \partial_\alpha j_i^\alpha(z_1) \int d^2 z_2 \partial_\alpha j_i^\alpha(z_2). \quad (2.9)$$

The dilute gas approximation is justified in a weak string coupling limit. As in the pointlike case, we assume the system is weakly coupled.

3 Scattering off the monopole

We saw that in the dilute gas approximation the complete effect of the zero modes (2.5) propagating in the wormhole can be summarized by the inclusion of a bilocal operator in the world-sheet action. One can rewrite this bilocal as a local term in the action by introducing wormhole parameters that are integrated over [5]:

$$\exp \left\{ \frac{1}{\kappa T} \left(4\pi \log \epsilon \int d^2 z \partial_\alpha j_i^\alpha \right)^2 \right\} = \int \prod_i d\alpha_i \exp \left\{ -\alpha_i^2 - 2\alpha_i \left(-\frac{4\pi \log \epsilon}{\sqrt{\kappa T}} \right) \int d^2 z \partial_\alpha j_i^\alpha \right\} \quad (3.1)$$

The wormhole parameters are like two dimensional θ parameters (that are integrated over), since they multiply total derivatives.

As will be shown below, the wormhole parameters α^i are related to the monopole's center of mass momentum. In other words one can interpret $\exp\{-\alpha^i N \int d^2z \partial_\alpha j_i^\alpha(z)\}$ as a “vertex operator” for the center of mass of the monopole with momentum proportional to $\vec{\alpha}$. In order to see this, consider the example of a matrix element corresponding to the elastic scattering of a string quantum off the monopole. The matrix element can be expressed as

$$\begin{aligned} \langle V_1 V_2 \rangle = & \int \prod_i d\alpha_i e^{-\alpha_i^2} \int Dx \exp \left\{ - \int d^2z \sum_j g_j \mathcal{O}_j \right\} \exp \left\{ \frac{8\pi\alpha^i \log \epsilon}{\sqrt{\kappa T}} \int d^2z \partial_\alpha j_i^\alpha \right\} \\ & \int d^2z_1 V_1(z_1) \int d^2z_2 V_2(z_2) \end{aligned} \quad (3.2)$$

The expression is evaluated on the sphere (to leading order). V_1 and V_2 are vertex operators, which can be written as $h(x)\mathcal{O}(\partial_\alpha x, \mathcal{J}_\alpha)$, where h are solutions of the linearized β -function equations. For large value of \vec{x} these vertex operators have the asymptotic behaviour

$$h(\vec{x}, t) \sim e^{i\vec{k} \cdot \vec{x} + i\delta_k} e^{-i\omega t}$$

where \vec{k} is a spatial momentum and δ_k is a phase shift. This is identical to the pointlike case; asymptotically (for large values of \vec{x}) the spatial momentum becomes a good quantum number, reflecting the fact that the monopole energy is concentrated in a finite region of space.

The scattering amplitude in (3.2) is computed with a new term in the action, proportional to the wormhole parameter $\vec{\alpha}$. The effects of this term can be easily evaluated. First, one needs to normal order this new term. This is easily shown to yield (Using results from the appendix) :

$$\exp \left\{ \frac{8\pi\alpha^i \log \epsilon}{\sqrt{\kappa T}} \int d^2z \partial_\alpha j_i^\alpha(z) \right\} = \exp \left\{ -\frac{8\pi\alpha_i^2 \log \epsilon}{T} \right\} : \exp \left\{ \frac{8\pi\alpha^i \log \epsilon}{\sqrt{\kappa T}} \int d^2z \partial_\alpha j_i^\alpha(z) \right\} : \quad (3.3)$$

Also, since j_i^α are the Nöether currents associated with spatial translations, $\exp \left\{ \int d^2 z \partial_\alpha j_i^\alpha(z) \right\}$ acts as a translation operator (This is easily verified using the Ward identity associated with translation). Therefore

$$\exp \left\{ \frac{8\pi\alpha^i \log \epsilon}{\sqrt{\kappa T}} \int d^2 z \partial_\alpha j_i^\alpha \right\} \prod_i V_i(x) = \exp \left\{ \frac{8\pi\alpha^i \log \epsilon}{\sqrt{\kappa T}} \frac{\partial}{\partial x} \right\} \prod_i V_i(x) \quad (3.4)$$

Finally, one has to contract the vertex operators among themselves. This can be done using the operator formalism on the sphere,

$$\langle V_1 V_2 \rangle = \langle V_1 | V_2 \rangle = \sum_a \langle V_1 | \phi_a \rangle \langle \phi_a | V_2 \rangle$$

The states $|\phi_a\rangle$ are defined in (2.3). The logarithmic dependence on the cut-off arises from the states annihilated by L_0 and \tilde{L}_0 , as discussed above. Therefore:

$$\langle V_1 V_2 \rangle = N^2 \langle V_1 | \partial_\alpha j_i^\alpha \rangle \langle \partial_\alpha j_i^\alpha | V_2 \rangle + \text{finite terms}$$

using the Ward identity mentioned above we obtain:

$$V_1 V_2 = - \sum_i \frac{4\pi \log \epsilon}{\kappa} \frac{\partial V_1}{\partial x_i} \frac{\partial V_2}{\partial x_i}. \quad (3.5)$$

The above results enable us to write the matrix element (3.2) as follows:

$$\begin{aligned} \langle V_1 V_2 \rangle &= \int \prod_i d\alpha_i e^{-\alpha_i^2} \exp \left\{ -\frac{\pi}{2} \log \epsilon \left(\frac{4\alpha_i}{\sqrt{T}} - \frac{2}{\sqrt{\kappa}} \frac{\partial}{\partial x} \right)^2 \right\} \\ &\quad \int D x e^{-I} \int d^2 z_1 V_1(z_1) \int d^2 z_2 V_2(z_2). \end{aligned} \quad (3.6)$$

Imposing conformal invariance and going to the continuum limit then forces the wormhole parameter to satisfy

$$\vec{\alpha} = \sqrt{\frac{T}{4\kappa}} (\vec{k}_1 + \vec{k}_2).$$

This relation is enforced through a $\delta^{(D)}$ $\left(\vec{\alpha} - \sqrt{\frac{T}{4\kappa}} (\vec{k}_1 + \vec{k}_2) \right)$. Therefore conformal invariance forces the monopole to recoil with the exact momentum needed to conserve the total momentum.

Performing the integration over $\vec{\alpha}$ results in the S-matrix element acquiring an additional phase shift $\exp -\frac{(\vec{k}_1+\vec{k}_2)^2 T}{4\kappa}$ due to the recoiling monopole. By Lorentz invariance in spacetime, the two-dimensional quantity κ (calculated on the sphere), is to be identified with half the classical mass of the monopole. We conjecture this relation between κ and the mass to hold for an arbitrary solitonic background.

One also expects the energy conservation condition to be modified to include the recoil of the monopole. We expect in this respect a violation of conformal invariance for the S-matrix element of the form

$$\exp \left\{ -\log \epsilon \left(\omega_1 + \omega_2 - \frac{(\vec{k}_1 + \vec{k}_2)^2}{4\kappa} \right)^2 \right\}.$$

Conformal invariance will then be restored by requiring the modified energy conservation condition. However, the modification due to the recoil will only appear in two loop order. Indeed, the term $\log \epsilon \frac{(\vec{k}_1+\vec{k}_2)^4}{16\kappa^2}$ appearing in the above expression can only be obtained through the dependence of the matrix elements on the wormhole parameters α^i raised to the fourth power.

4 Conclusions

We have seen that in the presence of solitons the two dimensional field theory displays non-local violations of conformal invariance. The wormhole parameters that are then introduced appear as two-dimensional θ parameters that are integrated over. There is one such parameter for each quantum mechanical degree of freedom associated to the recoil of the soliton. Requiring the theory to be ultimately conformally invariant, fixes the values of these wormhole parameters such that the soliton recoils with overall momentum conservation. Obviously all that has been said about recoil applies to charge deposition or any other change in the quantum numbers of the monopole. Similar non-localities

will appear on the worldsheet in the case of space-time instantons because of the existence of normalizable zero modes.

A more challenging situation occurs when describing multimonopole scattering. As was reviewed by Atiyah and Hitchin in the BPS limit [6] this scattering can be described by the motion of collective coordinates on an hyperkähler manifold. It would be interesting to understand how in this case the interaction between wormhole parameters appear from degenerating surfaces of higher genus such as to reproduce the known dynamics of the collective coordinates.

More generally the emergence of such non-local violations of conformal invariance will lead to the proper measure for summing over two-dimensional field theories which may be required in certain circumstances in string theory, like in the example discussed above.

5 Acknowledgements

We would like to acknowledge useful conversations with J. Russo at an earlier stage of this work. We would like also to thank V. Kaplunovsky, M. Dine, J. Polchinski and L. Susskind for discussions. This work is supported in part by the Robert A. Welch Foundation and NSF Grant PHY 9009850.

Appendix

The proper way to normalize states appearing in (2.2) was discussed in [4]. This leads to the following normalization condition for the operators (2.5) :

$$N^2 \int_{S^2} Dx Dc Db e^{-I} c_0(0) \bar{c}_0(0) \varphi_i(0) \varphi_j(z) = 4\pi \delta_{ij} \quad (\text{A.1})$$

The point z is arbitrary since in the special case of the operators (2.5) the expression is conformally invariant. Performing this functional integration

gives:

$$-\frac{1}{2} N^2 \kappa |z|^4 \left(\bar{\partial}^2 \frac{1}{z^2} + c \cdot c \right) = 4\pi \quad (\text{A.2})$$

where we have used:

$$\int_{S^2} Dx e^{-I} j_i(z) j_j(0) = \langle j_i(z) j_j(0) \rangle = \frac{\kappa \delta_{ij}}{z^2}. \quad (\text{A.3})$$

This integral is determined by it's conformal weight, up to the arbitrary constant κ .

The expression (A.2) is independent of the choice of z , so we can integrate over z (with the conformally invariant weight). This integration gives:

$$4\pi N^2 \kappa = 4\pi \int \frac{d^2 z}{|z|^2} = -16\pi^2 \log \epsilon + \text{finite}$$

or

$$N^2 = -\frac{4\pi \log \epsilon}{\kappa} + \text{finite}. \quad (\text{A.4})$$

References

- [1] R. Rajaraman, Solitons and Instantons, North-Holland, Amsterdam, 1982.
- [2] C. Rebbi and G. Soliani, Solitons and Particles, World Scientific, Singapore, 1984.
- [3] C.G. Callan, J.A. Harvey and A. Strominger, Supersymmetric String Solitons, in String Theory and Quantum Gravity 1991: Proceedings of the Trieste spring school. World Scientific, Singapore, 1991, and references therein.
- [4] J. Polchinski, Nucl. Phys. **B307**, (1988), 61.
- [5] S. Coleman, Nucl. Phys. **B310** (1988), 643.
- [6] M.F. Atiyah and N.J. Hitchin, The Geometry and Dynamics of Magnetic Monopoles, Princeton, 1988.

Figure Captions

Figure 1: Example of a surface with one handle.

Figure 2: The surface Σ .

This figure "fig1-1.png" is available in "png" format from:

<http://arXiv.org/ps/hep-th/9503072v1>

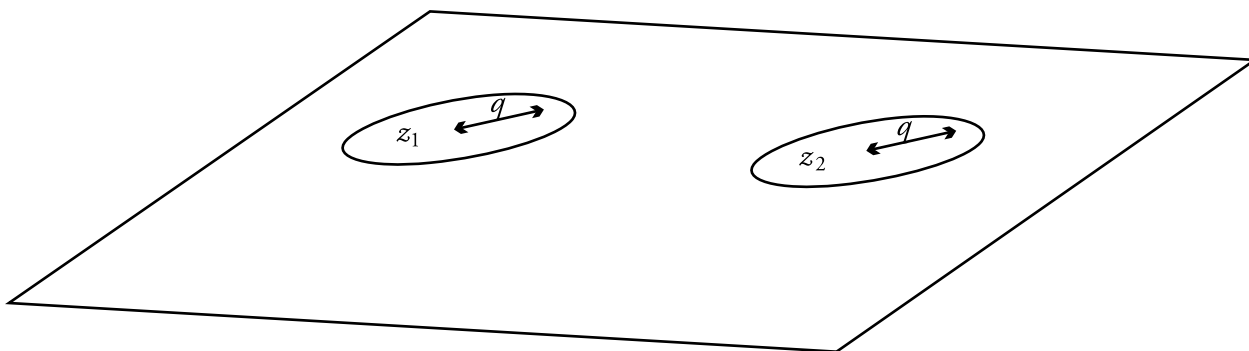


Figure 1

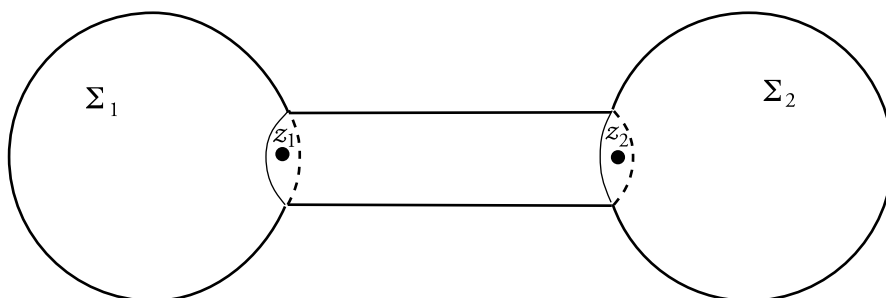


Figure 2